ACCELERATION OF CONVERGENCE OF THE FINITE-ELEMENT METHOD

E. Della Torre and W. Kinsner
Department of Electrical Engineering, McMaster University
Hamilton, Ontario, Canada

Abstract

An extrapolation technique has been found effective in reducing the computation time in solving field problems by the method of finite differences. This technique has been applied to the finite-element method with a corresponding reduction in computation time.

Summary

Extrapolation techniques have been useful in accelerating convergence when solving partial differential equations iteratively. It has been shown[1] that substantial reduction in computation time can be obtained if the sequence of solution vectors produced by the iterative method converges linearly. This has been the case for the solution of field problems by the finite-difference method[2,3].

The finite-element method is useful in solving field problems since it can drastically reduce the number of unknown variables in the problem when compared to the finite difference method. After the problem has been formulated iterative techniques are used if the number of variables is still large or if the problem is finally described by large systems of nonlinear algebraic and transcendental equations[4]. These are based on sparse matrix formulation. If the number of variables is sufficiently small conventional solutions are obtained[5]. We shall address ourselves to the former problem.

In this paper, the finite-element formulation of the problems is unchanged. Particularly, if one considers scalar potential problems formulated as boundary-value problems consisting of the inhomogeneous Helmholtz equation

$$\nabla^2\phi = g - \lambda^2\phi \tag{1}$$

subject to associated natural boundary conditions, one may solve (1) by minimizing the functional[6]

$$\mathcal{F} = \frac{1}{2}\int (|\nabla\phi|^2 + \lambda^2\phi^2)dS - \int\phi g dS \tag{2}$$

This minimization process within any subregion (an element) is described by the condition

$$\frac{\partial\mathcal{F}}{\partial\phi_n} = 0 \quad \forall n \tag{3}$$

which can be reformulated as the general matrix eigenvalue equation

$$S\underset{\sim}{\phi} = T(\underset{\sim}{G} - \lambda^2\underset{\sim}{\phi}) \tag{4}$$

where S and T are square, symmetric matrices whose elements are given by

$$S_{ij} = \int\int (\frac{\partial\alpha_i}{\partial x}\frac{\partial\alpha_j}{\partial x} + \frac{\partial\alpha_i}{\partial y}\frac{\partial\alpha_j}{\partial y})dxdy \tag{5a}$$

and

$$T_{ij} = \int\int \alpha_i\alpha_j \, dxdy \tag{5b}$$

while α_n are the polynomials of degree N in the coordinates. Assembly of the matrices S and T is simple, for they can be constructed out of invariant matrices[6] T_e and Q_i. The general eigenvalues problem (4) can be solved by any suitable technique[5]. The extrapolation technique is used to accelerate the convergence of the iteration process used to solve the linear equations. After every third iteration, a modification of Aitken's δ^2 process is used to extrapolate the convergent vector sequence to its limit. Extrapolation is improved if the rate of convergence of the original vector sequence is known.

The extrapolation formula used is given by

$$y_i^{(n)} = \begin{cases} x_i^{(n)} + 2(1+\gamma)\delta_i^{(n)}, & \delta_i^{(n-1)}\delta_i^{(n)} \geq 0 \text{ and } |\delta_i^{(n)}| \leq |\delta_i^{(n-1)}| \\ x_i^{(n)} + (3+2\gamma)\delta_i^{(n-1)}, & \delta_i^{(n-1)}\delta_i^{(n)} \geq 0 \text{ and } |\delta_i^{(n)}| > |\delta_i^{(n-1)}| \\ x_i^{(n)} & ,\delta_i^{(n-1)}\delta_i^{(n)} < 0 \end{cases} \tag{6}$$

where $\delta_i^{(n)}$ is the i-th element of the displacement vector $\underset{\sim}{\delta}^{(n)} = \underset{\sim}{x}^{(n)} - \underset{\sim}{x}^{(n-1)}$ at n-th iteration, and γ is the estimate of the rate of convergence of $\underset{\sim}{x}$ obtained using the formula

$$\gamma^{(n)} = \frac{||\underset{\sim}{\delta}^{(n)}||_1}{||\underset{\sim}{\delta}^{(n-1)}||_1} \tag{7}$$

where $||\underset{\sim}{\delta}^{(n)}||_1$ is the first norm of the displacement vector.

A ratio of the error after and before extrapolation is given by

$$R = \frac{f_i^{(n)}}{e_i^{(n)}} = \frac{2\sigma_i^{(n-1)} + \alpha_i(1+2\kappa) - 2\kappa}{\alpha_i + \sigma_i^{(n-1)}} \tag{8}$$

where it is assumed that the vector sequence converges linearly according to the formula given by

$$\underset{\sim}{x}^{(n)} = D^{(n)}\,\underset{\sim}{x}^{(n-1)} + \underset{\sim}{\varepsilon}\,, \tag{9}$$

where

$$D^{(n)} = \begin{bmatrix} \alpha_{11} + \sigma_{11}^{(n)} & & & \mathbf{0} \\ & \alpha_{22} + \sigma_{22} & & \\ & & \ddots & \\ \mathbf{0} & & & \alpha_{NN} + \sigma_{NN}^{(n)} \end{bmatrix} \tag{10}$$

where $\underset{\sim}{\varepsilon}$ is the error vector introduced by any numerical process, and $\kappa = \gamma+1$.

Fig. 1 is a graph of the ratio of error after and before extrapolation. A ratio of zero indicates that the answer is obtained correctly after extrapolation. A ratio less than one indicates improvement in the rate of convergence. The figure represents the process for $\kappa = 7/6$. It is seen that if the actual rate of convergence, α, is 0.7 and the random error σ is zero then the error ratio is zero. A ten-fold improvement in convergence results if the errors σ are less than 4% and the approximation to the real rate of convergence is within 5%.

This technique of accelerating convergence of the finite-element method has been applied to waveguide solutions previously discussed by Silvester[7].

References

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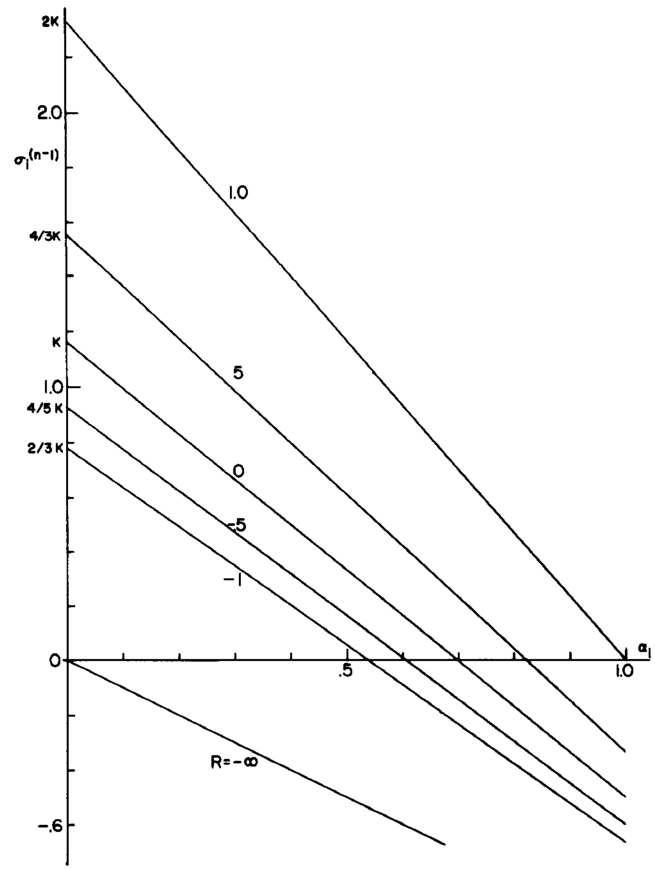

FIG. 1. Error ratio as a function of α_i and σ_i

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